The Grothendieck-Teichmüller group and the outer automorphism groups of the profinite braid groups (joint work with Hiroaki Nakamura)

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$n > 3$: an integer

B_n: the (Artin) braid group on n strings

Example: $(n = 4)$

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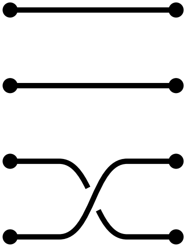

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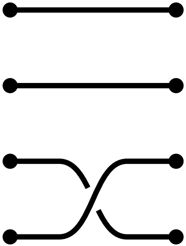

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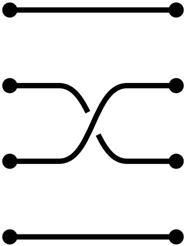

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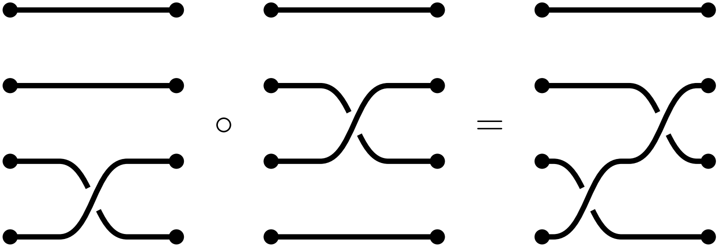

Note: We have

$$B_n \cong \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i \;=\; \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}; \\ \sigma_i \cdot \sigma_j \;=\; \sigma_j \cdot \sigma_i \;\; (|i-j| \;\geq\; 2) \end{array} \right\rangle$$

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Theorem (Dyer-Grossman)

The natural surjection $\mathrm{Aut}(B_n) \twoheadrightarrow \mathrm{Out}(B_n)$ *induces*

$$\langle \iota \rangle \xrightarrow{\sim} \mathrm{Out}(B_n).$$

Profinite case

$\widehat{B}_n$: the profinite completion of B_n

$\mathfrak{S}_n$: the symmetric group on n letters $\curvearrowright$

$(\mathbb{A}^1_{\mathbb{Q}})_n \stackrel{\text{def}}{=} \{(x_1, \ldots, x_n) \in (\mathbb{A}^1_{\mathbb{Q}})^n \mid x_i \neq x_j \ (i \neq j)\}$

The structure morphism $(\mathbb{A}^1_{\mathbb{Q}})_n/\mathfrak{S}_n \to \mathrm{Spec}(\mathbb{Q})$ induces

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The structure morphism $(\mathbb{A}^1_{\mathbb{Q}})_n/\mathfrak{S}_n \to \mathrm{Spec}(\mathbb{Q})$ induces

$$G_{\mathbb{Q}} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{Out}(\pi_1(((\mathbb{A}^1_{\mathbb{Q}})_n/\mathfrak{S}_n) \times_{\mathbb{Q}} \overline{\mathbb{Q}})) \cong \mathrm{Out}(\widehat{B}_n).$$

Drinfeld and Ihara defined a certain subgroup $\widehat{\mathrm{GT}} \subseteq \mathrm{Aut}(\widehat{\mathbb{Z} * \mathbb{Z}})$, called the (profinite) Grothendieck-Teichmüller group, such that there exists a commutative diagram

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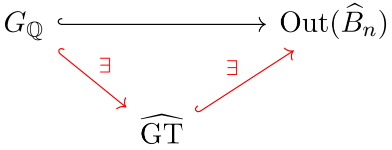

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$$\begin{array}{ccc} G_{\mathbb{Q}} & \hookrightarrow & \mathrm{Out}(\widehat{B}_n) \\ & {\scriptstyle \exists}\searrow \quad \nearrow{\scriptstyle \exists} & \\ & \widehat{\mathrm{GT}} & \end{array}$$

Open problem: Is $G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}$ an isomorphism?

Theorem (M.-Nakamura)

Write

$$Z_n \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\widehat{\mathbb{Z}}^\times \twoheadrightarrow (\widehat{\mathbb{Z}}/n(n-1)\widehat{\mathbb{Z}})^\times).$$

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Note: If $G_{\mathbb{Q}} \overset{\sim}{\to} \widehat{\mathrm{GT}}$, then we have $Z_n \times G_{\mathbb{Q}} \overset{\sim}{\to} \mathrm{Out}(\widehat{B}_n)$.

Definition of $Z_n \to \mathrm{Aut}(\widehat{B}_n)$ ($\twoheadrightarrow \mathrm{Out}(\widehat{B}_n)$)

Note: The center $C_n \subseteq B_n$ is an infinite cyclic group ($\cong \mathbb{Z}$) generated by $\zeta_n \stackrel{\text{def}}{=} (\sigma_1 \cdots \sigma_{n-1})^n$.

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$\implies$ We obtain a homomorphism $\phi_\nu : \widehat{B}_n \to \widehat{B}_n$.

Lemma 1

It holds that

$$\phi_1 = \mathrm{id}; \qquad \phi_{\nu_1 \cdot \nu_2} = \phi_{\nu_1} \circ \phi_{\nu_2} \quad (\nu_1, \nu_2 \in Z_n).$$

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$\Longrightarrow$ We obtain a homomorphism

$$\begin{aligned} \phi : Z_n &\to \mathrm{Aut}(\widehat{B}_n) \\ \nu &\mapsto \phi_\nu \end{aligned}$$

Outline of the proof of Theorem

Step1 Write $\mathcal{B}_n \stackrel{\mathrm{def}}{=} B_n/C_n$. We show that the composite

$$\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{B}_n) \to \mathrm{Out}(\widehat{\mathcal{B}}_n)$$

is an isomorphism. (Note that we have $\widehat{\mathcal{B}}_n \xrightarrow{\sim} \widehat{B}_n/\widehat{C}_n$.)

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Corollary (M.-Nakamura)

We have a natural isomorphism

$$\widehat{\mathrm{GT}}\ [\stackrel{\sim}{\rightarrow}\ \mathrm{Out}(\widehat{\mathcal{B}}_4)]\ \stackrel{\sim}{\rightarrow}\ \mathrm{Out}(\widehat{\Gamma}_{1,2}).$$

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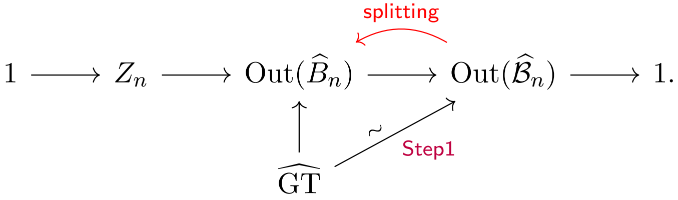

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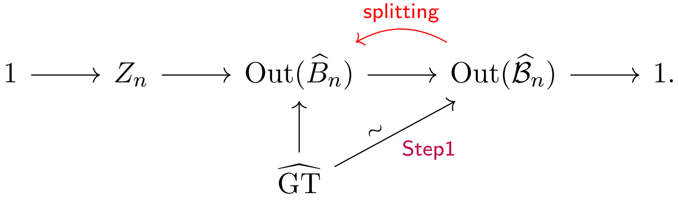

Therefore, we conclude that $Z_n \times \widehat{\mathrm{GT}} \xrightarrow{\sim} \mathrm{Out}(\widehat{B}_n)$.

Details of Step1

Idea Observe that $\mathcal{P}_n \stackrel{\text{def}}{=} \mathrm{Ker}(\mathcal{B}_n \twoheadrightarrow \mathfrak{S}_n)$ may be identified with

$\Gamma_{0,n+1}$: the pure mapping class group of sphere w/ $n+1$ marked pts.

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Theorem (Hoshi-M.-Mochizuki)

We have a natural isomorphism

$$\mathfrak{S}_{n+1} \times \widehat{\mathrm{GT}} \overset{\sim}{\to} \mathrm{Out}(\widehat{\Gamma}_{0,n+1}).$$

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$$\cong \left\langle \omega_1, \omega_2, \ldots, \omega_n \;\middle|\; \begin{array}{l} \text{"braid relations"};\ (\omega_1 \cdots \omega_n)^{n+1} = 1; \\ \omega_1 \cdots \omega_{n-1} \cdot \omega_n^2 \cdot \omega_{n-1} \cdots \omega_1 = 1 \end{array} \right\rangle.$$

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Note: We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\mathcal{P}}_n & \longrightarrow & \widehat{\mathcal{B}}_n & \longrightarrow & \mathfrak{S}_n & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \widehat{\Gamma}_{0,n+1} & \longrightarrow & \widehat{\Gamma}_{0,[n+1]} & \longrightarrow & \mathfrak{S}_{n+1} & \longrightarrow & 1. \end{array}$$

— where $\widehat{\mathcal{B}}_n \to \widehat{\Gamma}_{0,[n+1]}$ is defined to be $\overline{\sigma}_i \mapsto \omega_i$.

Lemma 2

Let

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an exact sequence of finitely generated profinite groups. Write $\rho : G \to \mathrm{Out}(\Delta)$ *for the outer rep'n assoc. to the exact sequence.*

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- Δ *and* G *are center-free.*
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- Δ *and* G *are center-free.*
- $\Delta \subseteq \Pi$ *is a characteristic subgroup.*

Then we have an exact sequence

$$1 \longrightarrow Z_{\mathrm{Out}(\Delta)}(\mathrm{Im}(\rho)) \longrightarrow \mathrm{Out}(\Pi) \longrightarrow \mathrm{Out}(G)$$

— *where* $Z_{\mathrm{Out}(\Delta)}(\mathrm{Im}(\rho))$ *is the centralizer of* $\mathrm{Im}(\rho)$ *in* $\mathrm{Out}(\Delta)$.

We would like to apply Lemma 2 to the exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\mathcal{P}}_n & \longrightarrow & \widehat{\mathcal{B}}_n & \longrightarrow & \mathfrak{S}_n & \longrightarrow & 1 \\ & & \uparrow \wr & & & & & & \\ & & \widehat{\Gamma}_{0,n+1} & & & & & & \end{array}$$

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to obtain an exact sequence

$$1 \longrightarrow Z_{\mathrm{Out}(\widehat{\Gamma}_{0,n+1})}(\mathfrak{S}_n) \longrightarrow \mathrm{Out}(\widehat{\mathcal{B}}_n) \longrightarrow \mathrm{Out}(\mathfrak{S}_n)$$

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Note: $\widehat{\mathcal{P}}_n \; [\cong \widehat{\Gamma}_{0,n+1}]$ and $\mathfrak{S}_n$ are center-free.

Thus, to apply Lemma 2, it suffices to check the following:

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Lemma 3

Let G *be a residually finite gp (i.e.,* $G \hookrightarrow \widehat{G}$*);* $N \subseteq G$ *a finite index normal subgp. Suppose that* $\mathrm{Ker}(G \overset{\forall}{\twoheadrightarrow} Q \overset{\mathrm{def}}{=} G/N)$ *coincides with* N*. Then* $\mathrm{Ker}(\widehat{G} \overset{\forall}{\twoheadrightarrow} Q)$ *coincides with* $\widehat{N}$*.*

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(a) $\varphi(\sigma_1) = (1, 2, 3, 4)$, $\varphi(\sigma_2) = (2, 1, 3, 4)$, $\varphi(\sigma_3) = (1, 2, 3, 4)$;

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Remark: In the proof of [DG, Theorem 11] claiming that $\mathcal{P}_4 \subseteq \mathcal{B}_4$ is characteristic, there is an inaccurate argument. They forgot to treat the case (b). (Moreover, the argument which was applied to "eliminate the case (a)" does not function properly for the case (b).)

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Remark: Using this argument, we can also prove the "centrality".